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Assignment 4

Due: Wednesday, November 1st, 2017, upload before 11:59pm

1. (50 pts.) Solve the recurrences:

a) an = 4an-1

a0=1

∀n > 1

an = 41 an-1

Substitute = 41 [4 an-2]

Simplify = 42 ∙ an-2

Substitute = 42 [4 an-3]

Simplify = 43 ∙ an-3

Etc…

= 4k ∙ an-k

k = n

4n ∙ an-n = 4n ∙ a0 = 4n ∙ 1

an = 4n ∀n > 1

Prove true for n = 1

a1 = 41 = 4 a1 = 4a1-1 = 4a0 = 4(1) = 4

True for n = 1

Assume true for n = k

ak = 4k

Consider n = k+1

ak+1 = 4ak

= 4(4k)

= 4k+1

If true for n = k, then also true for n = k+1

As true for n = 1, true for all positive numbers

b) an = can−1

a0=1

∀n > 1

c is some constant greater than 1

an = c1 an-1

Substitute = c1 [c an-2]

Simplify = c2 ∙ an-2

Substitute = c2 [c an-3]

Simplify = c3 ∙ an-3

Etc…

= ck ∙ an-k

k = n

cn ∙ an-n = cn ∙ a0 = cn ∙ 1

an = cn ∀n > 1

Prove true for n = 1

a1 = c1 = c a1 = ca1-1 = ca0 = c(1) = c

True for n = 1

Assume true for n = k

ak = ck

Consider n = k+1

ak+1 = cak

= c(ck)

= ck+1

If true for n = k, then also true for n = k+1

As true for n = 1, true for all positive numbers

c) bn = bn−1 +n/2

b0=0

∀n > 1

Substitute = [bn−2 + (n-1)/2] + n/2

Simplify = bn−2 + (n-1)/2 + n/2

Substitute = [bn−3 + (n-2)/2] + (n-1)/2 + n/2

Simplify = bn−3 + (n-2)/2 + (n-1)/2 + n/2

ETC…

= [bn-(k+1) + (n-k)/2] + … + (n-2)/2 + (n-1)/2 + n/2

= bn-(k+1) + (n-k)/2 + … + (n-2)/2 + (n-1)/2 + n/2

Let n-(k+1) = 0

n-k-1 = 0

n = k+1

So n-(k+1) = 0

n-k = 1

n-(k-1) = 2 etc…

So we get

b0 + 1 + 2+ 3 + …. (n-1) + n

= b0 + n(n+1)/4

= 0 + n(n+1)/4

= n(n+1)/4

= (n2+n)/4 ∀n > 1

Prove true for n = 1

b1 = (12+1)/4 = (2)/4 = 1/2 b1 = b1-1 + ½ = b0 + ½ = 0 + ½ = 1/2

True for n = 1

Assume true for n = k

bk = (k2+k)/4

Consider n = k+1

bk+1 = bk + (k+1)/2

= (k2+k)/4 + (k+1)/2

= (k2+3k+2)/4

If true for n = k, then also true for n = k+1

As true for n = 1, true for all positive numbers

d) cn = 4cn−1 + n

c0=0

∀n > 1

Substitute = 4[4cn-2 + (n-1)] + n

Simplify = 42cn-2 + 4(n-1) + n

Substitute = 42[4cn-3 + (n-2)] + 4(n-1) + n

Simplify = 43cn-3 + 4(n-2) + 4(n-1) + n

ETC…

= 4k[4cn-(k+1) + (n-k)] + … + 4(n-2) + 4(n-1) + n

= 4k+1cn-(k+1) + (n-k) + … + 4(n-2) + 4(n-1) + n

Let n-(k+1) = 0

n-k-1 = 0

n = k+1

So n-(k+1) = 0

n-k = 1

n-(k-1) = 2 etc…

So we get

c0 + 1 + 2+ 3 + …. (n-1) + n

= (4n+1 - 3n - 4)/9

Prove true for n = 1

c1 = (41+1 -3(1) -4)/9 = (16 – 3 – 4)/9 = 9/9 = 1 c1 = 4c1-1 + 1 = 4c0 + 1 = 4(0) + 1 = 1

True for n = 1

Assume true for n = k

ck = (4k+1 – 3k - 4)/9

Consider n = k+1

ck+1 = 4ck + (k+1)

= 4[(4k+1 – 3k – 4)/9] + (k+1)

= (4k+2 – 12k -16)/9 + (9k+9)/9

= (4k+2 – 3k -7)/9

If true for n = k, then also true for n = k+1

As true for n = 1, true for all positive numbers

e) tn = 10tn−1 - 21tn−2

t0=0

t1=1

∀n ≥ 2

tn - 10tn−1 + 21tn−2 = 0

r2 - 10r + 21 = 0

(r-3)(r-7) r = 3, 7

tn = α(3n) + β(7n)

t0=0 = α(30) + β(70) = α+β

t1=1 = α(31) + β(71) = 3α+7β

0 = -3α-3β

1 = 3α+7β

+

1 = 4β

β = 1/4

1 = 3α + 7(1/4)

α = -1/4

tn = (-1/4)(3n) + (1/4)(7n)

= (7n – 3n)/4

f) tn = -5tn−1 +14tn−2 + 3n

t0=0

t1=1

∀n ≥ 2

tn + 5tn-1 – 14tn-2 = 0

r2 – 5r – 14 = 0

(r+7)(r-2) = 0 r = -7, 2

tn + 5tn-1 – 14tn-2 = 3n

A3n + 5A3n-1 – 14A3n-2 = 3n

A + 5A3-1 – 14A3-2 = 1=

A + 5/3A – 14/9A = 1

9/9A + 15/9A – 14/9A = 1

10A = 1

A = 1/10

Guess: 1/10(3n)

α(-7)n + β(2)n + 1/10(3n)

t0= 0 = α(-7)0 + β(2)0 + 1/10(30) = α + β + 1/10

-1/10 = α + β = 2/10 = -2α - 2β

t1= 1 = α(-7)1 + β(2)1 + 1/10(31) = -7α + 2β + 3/10

7/10 = -7α + 2β

9/10 = -9α

9/10 = 9α

α = -1/10

-1/10 = -1/10 + β

β = 0

tn = -1/10(-7)n - 0(2)n + 1/10(3n)

= -(3n – (-7)n)/10

= ((-7)n – (3)n)/10

g) tn = 3tn−1 + n + 3n

t0=0

∀n ≥ 1

tn - 3tn-1 = 0

r – 3 = 0

r = 3

tn - 3tn-1 = n + 3n

(x-3)(x-1)2(x-3) = 0

r = 1,1,3,3

tn = α11n + α2n1n + β13n + β2n3n

t0 = α110 + α2(0)10 + β130 + β2(0)30 ⇾ α1 + β1 = 0

t1 = α111 + α2(1)11 + β131 + β2(1)31 ⇾ α1 + α2 + 3β1 + 3β2 = 4

t2 = α112 + α2(2)12 + β132 + β2(2)32 ⇾ α1 + 2α2 + 9β1 + 18β2 = 23

t3 = α113 + α2(3)13 + β133 + β2(3)33 ⇾ α1 + 3α2 + 27β1 + 81β2 = 99

Solve the system of equations by substituting variables and get:

α1 = -3/4, α2 = -1/2, β1 = 3/4, β2 = 1

tn = (-3/4)1n + (-1/2)n1n + (3/4)3n + (1)n3n

= (-3/4) + (-1/2)n + (3/4)3n + n3n

1. (20 pts.) Express the following solutions in Big-O notation

a) T(n) = 4T( n/2) + n

∀n > 1

a = 4, b = 2, d = 1

logba = 2

2 > 1

Θ(nlog24)

b) T(n)= 5T( n/2) + n2

∀n > 1

a = 5, b = 2. d = 2

logba = 2.32193

2.32193 > 2

Θ(nlog25)

The following are bonus questions each worth 10 points.

c) T(n) =4T2(n-1)

T(0)=1

∀n ≥ 1

d) T(n) =nT2(n/3)

T(1)=6

∀n > 1

**Note**: Please remember to complete your solution using proof by induction whenever your solution involves a guess.

Programming Questions: the following questions (3 and 4) are programming questions.

1. (20 pts.) Solve the recurrence relation for the “**Tower of Hanoi**” problem. The problem statement is as follows:

Given a stack of n disks arranged from largest on the bottom to smallest on top placed on a rod, together with two empty rods (totally 3 rods), the towers of Hanoi puzzle asks for the minimum number of moves required to move the stack from one rod to another, where moves are allowed only if they place smaller disks on top of larger disks.

Solve the relation and Plot a graph for the various values of n disks. Give appropriate justifications for your relation.

The first step is to move the (n-1)-disk tower to the spare peg; this takes T(n-1) moves. Then move the nth disk, taking 1 move. And finally move the (n-1)-disk tower again, this time on top of the nth disk, taking T(n-1) moves. This gives us our recurrence relation,

T(1) = 1

T(n) = 2T(n-1) + 1

Substitute: 2[2T(n-2) +1] + 1

Simplify: 22T(n-2) + 2 + 1

Substitute: 22[2T(n-3) +1] + 2 + 1

Simplify: 23T(n-3) + 4 + 2 + 1

ETC…

T(n) = 2kT(n-k) + 2k – 1

Since n is finite

lim T(n) k⇾n = 2n – 1

Runtime is: T(n) = O(2n)

1. (20 pts.) Derive a recurrence relation on Binary Search.

Solve the relation and Plot a graph for the various values of n. Give appropriate justifications for your relation.

B(1) = 1

B(n) = B(n/2) + 1

B(n/2) = B(n/4) + 1

B(n/4) = B(n/8) + 1

….

B(4) = B(2) + 1

B(2) = B(1) + 1

B(n) + B(n/2) + B(n/4) + …. + B(2) = B(n/2) + B(n/4) + B(n/8) + …. B(2) + B(1) + (1+1+…+1)

B(n) = B(1) + Log(n)

B(n) = 1 + Log(n)

Run Time is: B(n) = O(Log(n))

**Note**: The run time should be calculated for Problems 3 and 4. The graph should be plotted with time on the X axis and value of n on the Y axis.